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INTERFACE CRACKS IN ANISOTROPIC ELASTIC BIMATERIALS—A DECOMPOSITION PRINCIPLE

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Abstract—For a mismatched bimaterial with a single interface crack subject to a constant traction along the crack surface, the solution can be written explicitly, separated into oscillatory part and non-oscillatory part. The separation is shown to be related to the decomposition of the surface traction $\mathbf{t}_\Gamma$ into two components $\mathbf{t}_\Gamma^0$ and $\mathbf{t}_\Gamma^*$. $\mathbf{t}_\Gamma^0$ is in the direction of the right null vector of the $\hat{\mathbf{S}}$ matrix defined in the paper and $\mathbf{t}_\Gamma^*$ lies on the right eigenplane of $\hat{\mathbf{S}}$. The solution associated with $\mathbf{t}_\Gamma^0$ is non-oscillatory. It has the property that the traction along the interface is in the direction of the right null vector of $\hat{\mathbf{S}}$ while the crack opening displacement is in the direction of the left null vector of $\hat{\mathbf{S}}$. The solution associated with $\mathbf{t}_\Gamma^*$, on the other hand, is oscillatory. It has the property that the traction along the interface lies on the right eigenplane of $\hat{\mathbf{S}}$ while the crack opening displacement lies on the left eigenplane of $\hat{\mathbf{S}}$. The same decomposition and properties hold for multiple interface cracks with variable tractions prescribed on the crack surfaces.

1. INTRODUCTION

The problem of an interface crack in isotropic bimaterials was first studied by Williams (1959) and Erdogan (1963) for the semi-infinite crack and by England (1965), Erdogan (1965) and Rice and Sih (1965) for a finite crack. For an interface crack in anisotropic bimaterials, Gotoh (1967) studied the problem under the condition of plane stress which applies to monoclinic materials with the plane of symmetry at $x_3 = 0$. The problem of a finite interface crack in general anisotropic bimaterials was first investigated by Clements (1971) and Willis (1971). In recent years the elegant and powerful Stroh formalism for two-dimensional anisotropic elasticity has rekindled interests in the subject and many works have appeared such as Ting (1986), Bassani and Qu (1989), Qu and Bassani (1989), Tewary *et al.* (1989), Suo (1990), Wu (1990, 1991, in press), Ting (1990b), Hwu (in press), Li and Nemat-Nasser (1991), Gao *et al.* (in press) and Qu and Li (in press).

It is known that the solution for the displacement is in general oscillatory when the two materials in the bimaterial are mismatched. However, a mismatched bimaterial does not always produce oscillatory solutions. The oscillation in displacement depends not only on the mismatch parameter β but also on the prescribed traction $\mathbf{t}_\Gamma$ on the crack surface.

After presenting briefly the Stroh formalism for two-dimensional elasticity and certain identities needed for the subject in Section 2, we begin Section 3 by considering the solution for a crack in homogeneous anisotropic elastic materials. We then explore the applicability of the solution for a crack in a homogeneous medium to an interface crack in bimaterials in Section 4. It is shown that the non-oscillatory solution is valid for a mismatched bimaterial if the prescribed traction is in the direction of the null vector of $\mathbf{W}$. Section 5 discusses the stress singularities at an interface crack tip which depend on the matrix $\hat{\mathbf{S}}$. The three right eigenvectors of $\hat{\mathbf{S}}$ are best represented by a right null vector (which is identical to the null vector of $\mathbf{W}$) and a right eigenplane. The decomposition of the solution into oscillatory and non-oscillatory fields is achieved by decomposing the prescribed crack surface traction into $\mathbf{t}_\Gamma^0$, which is in the direction of the right null vector of $\hat{\mathbf{S}}$ and $\mathbf{t}_\Gamma^*$, which lies on the right eigenplane of $\hat{\mathbf{S}}$. The solution associated with $\mathbf{t}_\Gamma^0$ is non-oscillatory. It has the property that the traction along the interface is in the direction of the right null vector of $\hat{\mathbf{S}}$ while the crack opening displacement is in the direction of the left null vector of $\hat{\mathbf{S}}$. In Section 6 the solution associated with $\mathbf{t}_\Gamma^*$ is shown to be oscillatory. It has the property that the traction along the interface lies on the right eigenplane of $\hat{\mathbf{S}}$ while the crack opening displacement lies on the left eigenplane of $\hat{\mathbf{S}}$. Similar properties are observed for line forces and line dislocations in anisotropic media (Ting, 1990a). The last section examines the general case of multiple interface cracks with variable tractions prescribed on the crack surfaces. The same decomposition and properties are shown to hold for the general case.

2. THE STROH FORMALISM

In a fixed rectangular coordinate system x_i ($i = 1, 2, 3$) let u_i, σ_{ij} be, respectively, the displacement and stress in an anisotropic elastic material. The stress strain laws and the equations of equilibrium are

$$\sigma_{ij} = C_{ijks} u_{k,s}, \tag{1}$$

$$C_{ijks} u_{k,sj} = 0, \tag{2}$$

where a comma stands for differentiation, repeated indices imply summation and C_{ijks} are the elasticity constants which are assumed to be fully symmetric and positive definite. For two-dimensional deformations in which u_i depends on x_1, x_2 only, a general solution to (2) is, in matrix notation (Eshelby *et al.*, 1953; Stroh, 1958, 1962),

$$\mathbf{u} = \mathbf{a} f(z), \quad z = x_1 + p x_2. \tag{3}$$

In the above f is an arbitrary function of z, and p and $\mathbf{a}$ are determined by inserting (3) into (2). We have

$$\{\mathbf{Q} + p(\mathbf{R} + \mathbf{R}^{\mathrm{T}}) + p^2 \mathbf{T}\}\mathbf{a} = 0 \tag{4}$$

where the superscript T denotes the transpose and $\mathbf{Q}$, $\mathbf{R}$, $\mathbf{T}$ are 3×3 real matrices whose components are

$$Q_{ik} = C_{i1k1}, \quad R_{ik} = C_{i1k2}, \quad T_{ik} = C_{i2k2}.$$

The stresses obtained by substituting $(3)_1$ into (1) can be written in terms of the stress function $\boldsymbol{\phi}$ as

$$\sigma_{i1} = -\phi_{i,2}, \quad \sigma_{i2} = \phi_{i,1}, \tag{5}$$

in which

$$\boldsymbol{\phi} = \mathbf{b} f(z), \tag{6}$$

$$\mathbf{b} = (\mathbf{R}^{\mathrm{T}} + p\mathbf{T})\mathbf{a} = -\frac{1}{p}(\mathbf{Q} + p\mathbf{R})\mathbf{a}. \tag{7}$$

The second equality in (7) follows from (4). It suffices therefore to consider the stress function $\boldsymbol{\phi}$ because the stresses σ_{ij} can be obtained by differentiation.

There are six eigenvalues p from (4) which consist of three pairs of complex conjugates. If p_α, $\mathbf{a}_\alpha$ ($\alpha = 1, 2, \ldots, 6$) are the eigenvalues and the associated eigenvectors, we let

$$\operatorname{Im} p_\alpha > 0, \quad p_{\alpha+3} = \bar{p}_\alpha, \quad \mathbf{a}_{\alpha+3} = \bar{\mathbf{a}}_\alpha, \quad \mathbf{b}_{\alpha+3} = \bar{\mathbf{b}}_\alpha, \quad (\alpha = 1, 2, 3),$$

where Im stands for the imaginary part and the overbar denotes the complex conjugate. Assuming that p_α are distinct, the general solution for $\mathbf{u}$ and $\boldsymbol{\phi}$ obtained by superposing six solutions of the form (3) and (6) are

$$\mathbf{u} = \sum_{\alpha=1}^{3} \{\mathbf{a}_\alpha f_\alpha(z_\alpha) + \bar{\mathbf{a}}_\alpha f_{\alpha+3}(\bar{z}_\alpha)\},$$

$$\boldsymbol{\phi} = \sum_{\alpha=1}^{3} \{\mathbf{b}_\alpha f_\alpha(z_\alpha) + \bar{\mathbf{b}}_\alpha f_{\alpha+3}(\bar{z}_\alpha)\}. \tag{8}$$

In (8) $f_1, f_2, \ldots, f_6$ are arbitrary functions of their argument and

$$z_\alpha = x_1 + p_\alpha x_2.$$

In most applications f_α assume the same function form so that we may write

$$f_\alpha(z_\alpha) = q_\alpha f(z_\alpha),$$

$$f_{\alpha+3}(\bar{z}_\alpha) = \bar{q}_\alpha \bar{f}(\bar{z}_\alpha), \quad \alpha = 1,2,3,$$

where q_α are complex constants. The second equation is for obtaining real solutions for $\mathbf{u}$ and $\boldsymbol{\phi}$. Equations (8) can then be written as

$$\mathbf{u} = 2\,\mathrm{Re}\,\{\mathbf{A}\langle f(z)\rangle \mathbf{q}\},$$

$$\boldsymbol{\phi} = 2\,\mathrm{Re}\,\{\mathbf{B}\langle f(z)\rangle \mathbf{q}\}. \tag{9}$$

Here Re stands for the real part, $\mathbf{A}$, $\mathbf{B}$ are the 3×3 complex matrices defined by

$$\mathbf{A} = [\mathbf{a}_1, \mathbf{a}_2, \mathbf{a}_3], \quad \mathbf{B} = [\mathbf{b}_1, \mathbf{b}_2, \mathbf{b}_3],$$

and $\langle f(z)\rangle$ is the diagonal matrix

$$\langle f(z)\rangle = \mathrm{diag}\,[f(z_1), f(z_2), f(z_3)].$$

For a given problem all one has to do is to determine the unknown function $f(z)$ and the complex constant $\mathbf{q}$.

The eigenvectors $\mathbf{a}_\alpha$ and the associated vectors $\mathbf{b}_\alpha$ are not unique. When they are normalized by

$$2\mathbf{a}_\alpha \cdot \mathbf{b}_\alpha = 1, \quad (\alpha \text{ not summed}),$$

the three Barnett–Lothe tensors defined by

$$\mathbf{S} = i(2\mathbf{A}\mathbf{B}^{\mathrm{T}} - \mathbf{I}), \quad \mathbf{H} = 2i\mathbf{A}\mathbf{A}^{\mathrm{T}}, \quad \mathbf{L} = -2i\mathbf{B}\mathbf{B}^{\mathrm{T}}, \tag{10}$$

are real (Barnett and Lothe, 1973). It is clear that $\mathbf{H}$ and $\mathbf{L}$ are symmetric. It can be shown that they are positive definite (Chadwick and Smith, 1977; Gundersen *et al.*, 1987; Ting, 1988) and that $\mathbf{SH}$, $\mathbf{LS}$, $\mathbf{H}^{-1}\mathbf{S}$, $\mathbf{SL}^{-1}$ are antisymmetric. Moreover $\mathbf{S}$, $\mathbf{H}$, $\mathbf{L}$ are related by

$$\mathbf{HL} - \mathbf{SS} = \mathbf{I}.$$

Let $\langle p\rangle$ be the diagonal matrix:

$$\langle p\rangle = \mathrm{diag}\,[p_1, p_2, p_3]$$

and

$$\mathbf{N}_1 = -\mathbf{T}^{-1}\mathbf{R}^{\mathrm{T}}, \quad \mathbf{N}_2 = \mathbf{T}^{-1}, \quad \mathbf{N}_3 = \mathbf{R}\mathbf{T}^{-1}\mathbf{R}^{\mathrm{T}} - \mathbf{Q}.$$

It is shown in (Ting, 1988) that

$$\mathbf{B}\langle p\rangle\mathbf{B}^{-1} = \mathbf{G}_1 + i\mathbf{G}_3, \tag{11}$$

where

$$\mathbf{G}_1 = \mathbf{N}_1^T - \mathbf{N}_3\mathbf{S}\mathbf{L}^{-1}, \quad \mathbf{G}_3 = -\mathbf{N}_3\mathbf{L}^{-1}.$$

Identity (11) will be useful in the sequel. Another identity needed is

$$\mathbf{M}^{-1} = i\mathbf{A}\mathbf{B}^{-1} = i(\mathbf{A}\mathbf{B}^T)(\mathbf{B}\mathbf{B}^T)^{-1} = \mathbf{L}^{-1} - i\mathbf{S}\mathbf{L}^{-1} \tag{12}$$

in which the last equality is deduced by applying (10). $\mathbf{M}$ is the surface impedance tensor which is a positive definite Hermitian (Ingebrigtsen and Tonning, 1969; Lothe and Barnett, 1976; Chadwick and Smith, 1977).

3. A CRACK IN A HOMOGENEOUS MEDIUM

Consider a crack of length $2a$ located at $x_2 = 0$, $|x_1| < a$ in a homogeneous anisotropic elastic medium. A uniform traction $\mathbf{t}_\Gamma$ is applied at the upper crack surface and $-\mathbf{t}_\Gamma$ is applied at the lower crack surface. The stresses vanish at infinity. The boundary conditions for the stress function $\boldsymbol{\phi}$ are

$$\boldsymbol{\phi} = 0, \quad \text{as} \quad |\mathbf{x}| \to \infty \tag{13}$$

$$\boldsymbol{\phi} = -x_1\mathbf{t}_\Gamma, \quad \text{at} \quad x_2 = \pm 0, \quad |x_1| < a. \tag{14}$$

The solution in the form of (9) is (Stroh, 1958):

$$\begin{aligned}\mathbf{u} &= \mathrm{Re}\,\{\mathbf{A}\langle f_0(z)\rangle\mathbf{B}^{-1}\}\mathbf{t}_\Gamma,\\ \boldsymbol{\phi} &= \mathrm{Re}\,\{\mathbf{B}\langle f_0(z)\rangle\mathbf{B}^{-1}\}\mathbf{t}_\Gamma,\end{aligned} \tag{15}$$

where

$$f_0(z) = \sqrt{z^2 - a^2} - z. \tag{16}$$

For single-valuedness of the function $f_0(z)$, a cut at the crack is introduced so that

$$\sqrt{z_1^2 - a^2} = \begin{cases} \pm\sqrt{x_1^2 - a^2}, & \text{for } x_2 = 0, \quad \pm x_1 > a, \\ \pm i\sqrt{a^2 - x_1^2}, & \text{for } x_2 = \pm 0, \quad |x_1| < a. \end{cases} \tag{17}$$

It is readily shown that $\boldsymbol{\phi}$ of $(15)_2$ satisfies (13) and (14). Moreover, $\mathbf{u}$ and $\boldsymbol{\phi}$ are continuous everywhere except that $\mathbf{u}$ is discontinuous at the crack.

Along the x_1-axis, i.e. at $x_2 = 0$, the displacement $\mathbf{u}$ and the stresses obtained from (5) can be expressed in a real form. Denoting the traction vectors $\mathbf{t}_1$, $\mathbf{t}_2$ by

$$\mathbf{t}_1 = \begin{bmatrix}\sigma_{11}\\ \sigma_{21}\\ \sigma_{31}\end{bmatrix} = -\boldsymbol{\phi}_{,2}, \quad \mathbf{t}_2 = \begin{bmatrix}\sigma_{12}\\ \sigma_{22}\\ \sigma_{32}\end{bmatrix} = \boldsymbol{\phi}_{,1}, \tag{18}$$

and using identities (11) and (12), one obtains

$$\mathbf{u} = \{x_1 \mp \sqrt{x_1^2 - a^2}\}\mathbf{S}\mathbf{L}^{-1}\mathbf{t}_\Gamma,$$

$$\mathbf{t}_1 = \left\{\frac{x_1}{\pm\sqrt{x_1^2 - a^2}} - 1\right\}\mathbf{G}_1\mathbf{t}_\Gamma,$$

$$\mathbf{t}_2 = \left\{\frac{x_1}{\pm\sqrt{x_1^2-a^2}} - 1\right\}\mathbf{t}_\Gamma, \tag{19}$$

for $x_2 = 0$, $\pm x_1 > a$ and

$$\mathbf{u} = \pm\sqrt{a^2-x_1^2}\,\mathbf{L}^{-1}\mathbf{t}_\Gamma + x_1\mathbf{S}\mathbf{L}^{-1}\mathbf{t}_\Gamma,$$

$$\mathbf{t}_1 = \mathbf{G}_1\mathbf{t}_\Gamma \mp \frac{x_1}{\sqrt{a^2-x_1^2}}\mathbf{G}_3\mathbf{t}_\Gamma,$$

$$\mathbf{t}_2 = -\mathbf{t}_\Gamma \tag{20}$$

for $x_2 = \pm 0$, $|x_1| < a$. Equations (19) tell us that $\mathbf{u}, \mathbf{t}_1, \mathbf{t}_2$ along the x_1-axis are monotonous. The traction $\mathbf{t}_2$ is independent of the material constants and is in the direction of the applied traction $\mathbf{t}_\Gamma$. Equations (20) show that, along the crack, the displacement $\mathbf{u}$ and the hoop stress vector $\mathbf{t}_1$ have no oscillatory behavior. Other interesting and unexpected phenomena which can be extracted from (19) and (20) are elaborated in Ting (in press).

From $(20)_1$ the crack opening is

$$\Delta\mathbf{u} = \mathbf{u}^+ - \mathbf{u}^- = 2\sqrt{a^2-x_1^2}\,\mathbf{L}^{-1}\mathbf{t}_\Gamma.$$

The crack opening $\Delta\mathbf{u}$ is in general not in the direction of $\mathbf{t}_\Gamma$.

4. NON-OSCILLATORY SOLUTION FOR AN INTERFACE CRACK ($\mathbf{W}\mathbf{t}_\Gamma = 0$)

Let the half-space $x_2 > 0$ and $x_2 \leq 0$ be occupied by material 1 and material 2, respectively. A single interface crack is located at $x_2 = 0$, $|x_1| < a$. The boundary conditions for the problem are

$$\boldsymbol{\phi}_1 = \mathbf{0}, \quad \boldsymbol{\phi}_2 = \mathbf{0}, \quad \text{as} \quad |\mathbf{x}| \to \infty, \tag{21}$$

$$\mathbf{u}_1 = \mathbf{u}_2, \quad \boldsymbol{\phi}_1 = \boldsymbol{\phi}_2, \quad \text{at} \quad x_2 = 0, \quad |x_1| > a, \tag{22}$$

$$\boldsymbol{\phi}_1 = \boldsymbol{\phi}_2 = -x_1\mathbf{t}_\Gamma, \quad \text{at} \quad x_2 = \pm 0, \quad |x_1| < a. \tag{23}$$

The subscripts 1, 2 for $\mathbf{u}$ and $\boldsymbol{\phi}$ denote materials 1 and 2, respectively. We will investigate in this section if the non-oscillatory solution (15) for a homogeneous medium applies to materials 1 and 2.

Using subscripts 1, 2 or superscripts (1), (2) to distinguish materials 1 and 2, let

$$\mathbf{u}_1 = \mathrm{Re}\,\{\mathbf{A}_1\langle f_0(z^{(1)})\rangle\mathbf{B}_1^{-1}\}\mathbf{t}_\Gamma,$$

$$\boldsymbol{\phi}_1 = \mathrm{Re}\,\{\mathbf{B}_1\langle f_0(z^{(1)})\rangle\mathbf{B}_1^{-1}\}\mathbf{t}_\Gamma, \tag{24a}$$

for material 1 in $x_2 > 0$ and

$$\mathbf{u}_2 = \mathrm{Re}\,\{\mathbf{A}_2\langle f_0(z^{(2)})\rangle\mathbf{B}_2^{-1}\}\mathbf{t}_\Gamma,$$

$$\boldsymbol{\phi}_2 = \mathrm{Re}\,\{\mathbf{B}_2\langle f_0(z^{(2)})\rangle\mathbf{B}_2^{-1}\}\mathbf{t}_\Gamma, \tag{24b}$$

for material 2 in $x_2 < 0$. It is readily shown that conditions (21)–(23) are all satisfied except $(22)_1$, which yields

$$(\mathbf{A}_1\mathbf{B}_1^{-1} + \bar{\mathbf{A}}_1\bar{\mathbf{B}}_1^{-1})\mathbf{t}_\Gamma = (\mathbf{A}_2\mathbf{B}_2^{-1} + \bar{\mathbf{A}}_2\bar{\mathbf{B}}_2^{-1})\mathbf{t}_\Gamma.$$

Employing identity (12) this is rewritten as

$$\mathbf{W}\mathbf{t}_\Gamma = 0, \tag{25}$$

$$\mathbf{W} = \mathbf{S}_1\mathbf{L}_1^{-1} - \mathbf{S}_2\mathbf{L}_2^{-1}. \tag{26}$$

There are two possibilities. If $\mathbf{W} = 0$, (25) is satisfied for any $\mathbf{t}_\Gamma$. The solutions (24a,b) are then valid for any $\mathbf{t}_\Gamma$ when $\mathbf{W} = \mathbf{0}$. If $\mathbf{W} \neq \mathbf{0}$, (25) cannot be satisfied for arbitrary $\mathbf{t}_\Gamma$. However, since $\mathbf{W}$ is antisymmetric, (25) can still be satisfied if $\mathbf{t}_\Gamma$ is in the direction of the null vector of $\mathbf{W}$.

The two materials in the bimaterial are said to be "mismatched" when $\mathbf{W} \neq \mathbf{0}$. We see from the above analysis that, for a mismatched bimaterial, the non-oscillatory solution for a homogeneous medium with a crack applies to the bimaterial with an interface crack if the traction $\mathbf{t}_\Gamma$ at the crack surface is in the direction of the null vector of $\mathbf{W}$.

Regardless of whether $\mathbf{W} = \mathbf{0}$ or not, the crack opening $\Delta\mathbf{u}$ obtained from $(20)_1$ is

$$\Delta\mathbf{u} = \mathbf{u}_1^+ - \mathbf{u}_2^- = \sqrt{a^2 - x_1^2}\,\mathbf{D}\mathbf{t}_\Gamma, \tag{27}$$

where

$$\mathbf{D} = \mathbf{L}_1^{-1} + \mathbf{L}_2^{-1}. \tag{28}$$

When $\mathbf{W} \neq 0$ and if $\mathbf{t}_\Gamma$ is not in the direction of the null vector of $\mathbf{W}$, one could decompose $\mathbf{t}_\Gamma$ into two components, one of which is along the null vector of $\mathbf{W}$. The solution associated with this component is given by (24a,b). In the next two sections we define the other component and the corresponding solution which is characterized by an oscillatory field.

5. THE NULL VECTORS AND THE EIGENPLANES OF $\hat{\mathbf{S}}$

The stress singularities near the tip of an interface crack is proportional to r^δ where r is the radial distance from the crack tip and δ is a constant depending on the material property of the bimaterial. It is shown in Ting (1986) that there are three singularities given by

$$\delta = -\tfrac{1}{2}, \quad -\tfrac{1}{2}+i\gamma, \quad \text{and} \quad -\tfrac{1}{2}-i\gamma, \tag{29}$$

where

$$\gamma = \frac{1}{2\pi}\ln\frac{1+\beta}{1-\beta} = \frac{1}{\pi}\tanh^{-1}\beta$$

$$\beta = [-\tfrac{1}{2}\operatorname{tr}(\hat{\mathbf{S}}^2)]^{1/2} < 1. \tag{30}$$

In the above

$$\hat{\mathbf{S}} = \mathbf{D}^{-1}\mathbf{W}, \tag{31}$$

in which $\mathbf{D}$ and $\mathbf{W}$ are defined in (28) and (26). It is clear that $\gamma = 0$ if and only if $\beta = 0$. It was pointed out by Ting (1986) and rigorously proved by Qu and Bassani (1989) that $\beta = 0$ if and only if $\mathbf{W} = 0$. Since $\mathbf{D}$ is positive definite, we conclude that β, γ, $\mathbf{W}$ and $\hat{\mathbf{S}}$ are all non-zero for mismatched bimaterials and all vanish for non-mismatched bimaterials. In the rest of the paper we consider only mismatched bimaterials.

The tensor $\hat{\mathbf{S}}$ is similar to $\mathbf{S}$, one of three Barnett–Lothe tensors in the following sense. We see from (31) that $\hat{\mathbf{S}}$ is the product of a symmetric positive definite tensor $\mathbf{D}^{-1}$ and an antisymmetric tensor $\mathbf{W}$. So is $\mathbf{S}$ if we write

$$\mathbf{S} = \mathbf{L}^{-1}(\mathbf{LS}).$$

Another reason for using the notation $\hat{\mathbf{S}}$ is that, when $\mathbf{L}_1 = \mathbf{L}_2$, (31) reduces to

$$\hat{\mathbf{S}} = \tfrac{1}{2}(\mathbf{S}_2 - \mathbf{S}_1)^T.$$

The eigenvalues and eigenvectors of S have been studied extensively in Chadwick and Ting (1987) and Ting (1990a), and the results reported there can be applied to $\hat{\mathbf{S}}$ here. It is shown in Ting (1986) that the eigenvalues of $\hat{\mathbf{S}}$ are $-i\beta$, $i\beta$ and 0 where β is given in $(30)_3$. If the associated eigenvectors are $\mathbf{d}$, $\bar{\mathbf{d}}$ and $\mathbf{d}_0$, we have

$$\hat{\mathbf{S}}\mathbf{d} = -i\beta\mathbf{d}, \quad \hat{\mathbf{S}}\bar{\mathbf{d}} = i\beta\bar{\mathbf{d}}, \quad \hat{\mathbf{S}}\mathbf{d}_0 = \mathbf{0}. \tag{32}$$

$\mathbf{d}_0$ is the right null vector of $\hat{\mathbf{S}}$ and is a real vector. $\mathbf{d}$ on the other hand is a complex vector or a bivector (Gibbs, 1961; Boulanger and Hayes, in press). By setting

$$\mathbf{d} = \mathbf{d}_1 + i\mathbf{d}_2 \tag{33}$$

where $\mathbf{d}_1$, $\mathbf{d}_2$ are real vectors and equating the real and imaginary parts of $(32)_1$ we have

$$\hat{\mathbf{S}}\mathbf{d}_1 = \beta\mathbf{d}_2, \quad \hat{\mathbf{S}}\mathbf{d}_2 = -\beta\mathbf{d}_1. \tag{34}$$

Therefore

$$\hat{\mathbf{S}}^2\mathbf{d}_j = -\beta^2\mathbf{d}_j, \quad (j = 1, 2). \tag{35}$$

The right null vector $\mathbf{d}_0$ is unique up-to an arbitrary real multiplicative factor. The right eigenvector $\mathbf{d}$ or $\bar{\mathbf{d}}$ on the other hand is unique up to an arbitrary complex multiplicative factor. If $\mathbf{d}$ is multiplied by a complex factor $e^{i\psi}$ where ψ is real,

$$e^{i\psi}\mathbf{d} = \mathbf{d}'_1 + i\mathbf{d}'_2,$$

$$\mathbf{d}'_1 = \cos\psi\mathbf{d}_1 - \sin\psi\mathbf{d}_2, \quad \mathbf{d}'_2 = \sin\psi\mathbf{d}_1 + \cos\psi\mathbf{d}_2.$$

Thus $\mathbf{d}'_1$, $\mathbf{d}'_2$ lie on the plane spanned by $\mathbf{d}_1$, $\mathbf{d}_2$. As ψ varies the vectors $\mathbf{d}'_1$, $\mathbf{d}'_2$ describe an ellipse (Fig. 1). A pair of diameters in an ellipse is said to be conjugate if all chords parallel to one diameter are bisected by the other diameter. Therefore the tangent at the extremity of one diameter is parallel to the other diameter. It can be shown that $\mathbf{d}'_1$, $\mathbf{d}'_2$ form a pair of conjugate radii. One could choose a ψ such that $\mathbf{d}'_1$, $\mathbf{d}'_2$ are orthogonal and hence are the principal radii of the ellipse (Ting, 1990a).

In view of the fact that the real and imaginary parts of the right eigenvectors $\mathbf{d}$ and $\bar{\mathbf{d}}$ of $(32)_{1,2}$ lie on a plane, we call this plane the right eigenplane of $\hat{\mathbf{S}}$. Any vector on this

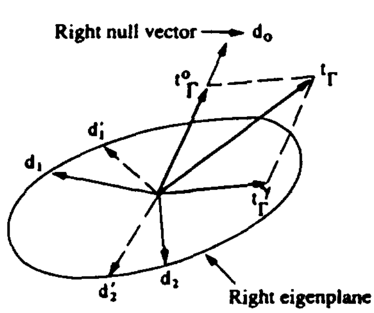

Fig. 1. The right eigenvectors of $\hat{\mathbf{S}}$.

plane can be taken as $\mathbf{d}_1$ (or $\mathbf{d}_2$), which satisfies (35). The conjugate vector $\mathbf{d}_2$ (or $\mathbf{d}_1$) is then determined from $(34)_1$ or $(34)_2$.

It should be pointed out that, with (31), the right null vector $\mathbf{d}_0$ is also the null vector of $\mathbf{W}$. Therefore, when $\mathbf{t}_\Gamma$ is in the direction of $\mathbf{d}_0$, the solution is given by (24a,b). If $\mathbf{t}_\Gamma$ is not in the direction of $\mathbf{d}_0$ as shown in Fig. 1, we decompose it into two components:

$$\mathbf{t}_\Gamma = \mathbf{t}_\Gamma^0 + \mathbf{t}_\Gamma^*, \tag{36}$$

where $\mathbf{t}_\Gamma^0$ is in the direction of $\mathbf{d}_0$ and $\mathbf{t}_\Gamma^*$ is on the right eigenplane. From $(32)_3$ and (35),

$$\hat{\mathbf{S}}\mathbf{t}_\Gamma^0 = \mathbf{0}, \quad \hat{\mathbf{S}}^2\mathbf{t}_\Gamma^* = -\beta^2\mathbf{t}_\Gamma^*. \tag{37}$$

Multiplying (36) by $\hat{\mathbf{S}}^2$ and using (37) leads to

$$\mathbf{t}_\Gamma^* = -\frac{1}{\beta^2}\hat{\mathbf{S}}^2\mathbf{t}_\Gamma, \tag{38}$$

whence, from (36),

$$\mathbf{t}_\Gamma^0 = \mathbf{t}_\Gamma + \frac{1}{\beta^2}\hat{\mathbf{S}}^2\mathbf{t}_\Gamma. \tag{39}$$

Equations (38) and (39) provide an explicit expression for $\mathbf{t}_\Gamma^0$ and $\mathbf{t}_\Gamma^*$ in terms of $\mathbf{t}_\Gamma$. The solution associated with $\mathbf{t}_\Gamma^0$ is given in (24a,b) with $\mathbf{t}_\Gamma$ there replaced by $\mathbf{t}_\Gamma^0$. We discuss the solution associated with $\mathbf{t}_\Gamma^*$ in the next section.

Before we close this section consider the left eigenvectors of $\hat{\mathbf{S}}$. From (31),

$$\mathbf{D}\hat{\mathbf{S}} = \mathbf{W},$$

and the antisymmetric property of $\mathbf{W}$ means that

$$\mathbf{D}\hat{\mathbf{S}} = -\hat{\mathbf{S}}^T\mathbf{D}. \tag{40}$$

When (32) are multiplied by $\mathbf{D}$ and use is made of (40),

$$\hat{\mathbf{S}}^T(\mathbf{Dd}) = i\beta(\mathbf{Dd}), \quad \hat{\mathbf{S}}^T(\mathbf{D\bar{d}}) = -i\beta(\mathbf{D\bar{d}}), \quad \hat{\mathbf{S}}^T(\mathbf{Dd}_0) = \mathbf{0}.$$

Hence $\mathbf{D\bar{d}}$, $\mathbf{Dd}$ and $\mathbf{Dd}_0$ are the left eigenvectors associated with the eigenvalues $-i\beta$, $i\beta$ and 0, respectively. We call $\mathbf{Dd}_0$ the left null vector of $\hat{\mathbf{S}}$ and the plane spanned by $\mathbf{Dd}_1$, $\mathbf{Dd}_2$ the left eigenplane (Fig. 2).

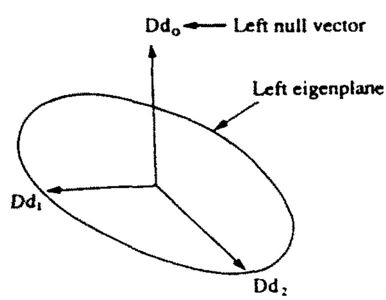

Fig. 2. The left eigenvectors of $\hat{\mathbf{S}}$.

The left and right eigenvectors associated with different eigenvalues are orthogonal to each other. Hence

$$\mathbf{d}^T\mathbf{D}\mathbf{d} = 0 = \bar{\mathbf{d}}^T\mathbf{D}\bar{\mathbf{d}}, \quad \mathbf{d}^T\mathbf{D}\mathbf{d}_0 = 0 = \bar{\mathbf{d}}^T\mathbf{D}\mathbf{d}_0. \tag{41}$$

The last equality tells us that the right null vector $\mathbf{d}_0$ is normal to the left eigenplane and the left null vector $\mathbf{D}\mathbf{d}_0$ is normal to the right eigenplane.

When $\mathbf{t}_\Gamma$ is in the direction of the right null vector $\mathbf{d}_0$, the crack opening displacement $\Delta\mathbf{u}$ given by (27) indicates that it is in the direction of the left null vector $\mathbf{D}\mathbf{d}_0$. The direction of the traction $\mathbf{t}_2$ along the interface is, by $(19)_3$, the same as $\mathbf{t}_\Gamma$ and hence is in the direction of the right null vector.

6. OSCILLATORY SOLUTION FOR AN INTERFACE CRACK ($\mathbf{W}\mathbf{t}_\Gamma \neq \mathbf{0}$)

In view of (31),

$$\mathbf{W}\mathbf{t}_\Gamma \neq \mathbf{0} \quad \text{and} \quad \hat{\mathbf{S}}\mathbf{t}_\Gamma \neq \mathbf{0}$$

are equivalent. As shown in (36) we decompose $\mathbf{t}_\Gamma$ into two components $\mathbf{t}_\Gamma^0$ and $\mathbf{t}_\Gamma^\gamma$ which are given explicitly in (39) and (38). The solution associated with $\mathbf{t}_\Gamma^0$ is provided in (24a,b) with $\mathbf{t}_\Gamma$ there replaced by $\mathbf{t}_\Gamma^0$. We now study the solution associated with $\mathbf{t}_\Gamma^\gamma$.

Of the three stress singularities at the crack tips listed in (29), the singularity $\delta = -1/2$ represents a non-oscillatory solution and is provided by the function $f_0(z)$ in (16). The other two singularities in (29) can be taken care of by considering the function

$$f(z,\gamma) = (z-a)^{1/2+i\gamma}(z+a)^{1/2-i\gamma} - z,$$

which has the property

$$\bar{f}(z,\gamma) = f(z,-\gamma).$$

Along the x_1-axis,

$$f(z_*,\gamma) = \begin{cases} \pm\sqrt{x_1^2-a^2}\,e^{i\gamma\xi} - x_1, & \text{for } x_2 = 0,\ \pm x_1 > a, \\ \pm i\sqrt{a^2-x_1^2}\,e^{\mp\gamma\pi}\,e^{i\gamma\xi} - x_1, & \text{for } x_2 = \pm 0,\ |x_1| < a, \end{cases} \tag{42}$$

where

$$\xi = \ln\left|\frac{x_1-a}{x_1+a}\right|.$$

Observing the factor $e^{\mp\gamma\pi}$ in $(42)_2$, consider the solution

$$\begin{aligned} \mathbf{u}_1 &= \mathrm{Re}\,\{e^{\gamma\pi}\mathbf{A}_1\langle f(z^{(1)},\gamma)\rangle\mathbf{B}_1^{-1}\mathbf{d} + e^{-\gamma\pi}\mathbf{A}_1\langle\bar{f}(z^{(1)},\gamma)\rangle\mathbf{B}_1^{-1}\bar{\mathbf{d}}\}, \\ \boldsymbol{\phi}_1 &= \mathrm{Re}\,\{e^{\gamma\pi}\mathbf{B}_1\langle f(z^{(1)},\gamma)\rangle\mathbf{B}_1^{-1}\mathbf{d} + e^{-\gamma\pi}\mathbf{B}_1\langle\bar{f}(z^{(1)},\gamma)\rangle\mathbf{B}_1^{-1}\bar{\mathbf{d}}\}, \end{aligned} \tag{43a}$$

for material 1 in $x_2 > 0$ and

$$\begin{aligned} \mathbf{u}_2 &= \mathrm{Re}\,\{e^{-\gamma\pi}\mathbf{A}_2\langle f(z^{(2)},\gamma)\rangle\mathbf{B}_2^{-1}\mathbf{d} + e^{\gamma\pi}\mathbf{A}_2\langle\bar{f}(z^{(2)},\gamma)\rangle\mathbf{B}_2^{-1}\bar{\mathbf{d}}\}, \\ \boldsymbol{\phi}_2 &= \mathrm{Re}\,\{e^{-\gamma\pi}\mathbf{B}_2\langle f(z^{(2)},\gamma)\rangle\mathbf{B}_2^{-1}\mathbf{d} + e^{\gamma\pi}\mathbf{B}_2\langle\bar{f}(z^{(2)},\gamma)\rangle\mathbf{B}_2^{-1}\bar{\mathbf{d}}\}, \end{aligned} \tag{43b}$$

for material 2 in $x_2 < 0$. The solution in material 1 is identical to that in material 2 if π is replaced by $-\pi$ (Suo, 1990). It can be shown that the solution meets conditions (21)–(23) with $\mathbf{t}_\Gamma$ in (23) replaced by $\mathbf{t}_\Gamma^\infty$ if $\mathbf{d}$, $\bar{\mathbf{d}}$ satisfy $(32)_{1,2}$ and

$$\mathbf{d}+\bar{\mathbf{d}} = \sqrt{1-\beta^2}\,\mathbf{t}_\Gamma^\infty. \tag{44}$$

To determine $\mathbf{d}$, (44) is multiplied by $\hat{\mathbf{S}}$, which leads to

$$-i\beta(\mathbf{d}-\bar{\mathbf{d}}) = \sqrt{1-\beta^2}\,\hat{\mathbf{S}}\mathbf{t}_\Gamma^\infty. \tag{45}$$

Equations (44) and (45) yield

$$\mathbf{d} = \frac{\sqrt{1-\beta^2}}{2\beta}(\beta\mathbf{t}_\Gamma^\infty + i\hat{\mathbf{S}}\mathbf{t}_\Gamma^\infty).$$

Real form expressions for the displacement $\mathbf{u}$, the interface traction $\mathbf{t}_2$ and the hoop stress vector $\mathbf{t}_1$ along the x_1-axis are deduced from (43a,b) by using (18), (11) and (12). For $x_2 = 0$, $\pm x_1 > a$,

$$\begin{aligned}
\mathbf{u}_1 &= 2(\cosh\gamma\pi)\,\mathrm{Re}\,\{(x_1 \mp e^{i\gamma\xi}\sqrt{x_1^2-a^2})(\mathbf{S}_1\mathbf{L}_1^{-1}+i\beta\mathbf{L}_1^{-1})\mathbf{d}\} = \\
\mathbf{u}_2 &= 2(\cosh\gamma\pi)\,\mathrm{Re}\,\{(x_1 \mp e^{i\gamma\xi}\sqrt{x_1^2-a^2})(\mathbf{S}_2\mathbf{L}_2^{-1}-i\beta\mathbf{L}_2^{-1})\mathbf{d}\},\\
\mathbf{t}_1^{(1)} &= -2(\cosh\gamma\pi)\,\mathrm{Re}\,\{[\pm\eta(x_1)\,e^{i\gamma\xi}-1][\mathbf{G}_1^{(1)}+i\beta\mathbf{G}_3^{(1)}]\mathbf{d}\},\\
\mathbf{t}_1^{(2)} &= -2(\cosh\gamma\pi)\,\mathrm{Re}\,\{[\pm\eta(x_1)\,e^{i\gamma\xi}-1][\mathbf{G}_1^{(2)}-i\beta\mathbf{G}_3^{(2)}]\mathbf{d}\},\\
\mathbf{t}_2 &\equiv 2(\cosh\gamma\pi)\,\mathrm{Re}\,\{[\pm\eta(x_1)\,e^{i\gamma\xi}-1]\mathbf{d}\},
\end{aligned} \tag{46}$$

where

$$\cosh\gamma\pi = \frac{1}{\sqrt{1-\beta^2}}, \quad \eta(x_1) = \frac{x_1+2i\gamma a}{\sqrt{x_1^2-a^2}}.$$

For $x_2 = \pm 0$, $|x_1| < a$,

$$\begin{aligned}
\mathbf{u}_1 &= 2\sqrt{a^2-x_1^2}\,\mathbf{L}_1^{-1}\,\mathrm{Re}\,\{e^{i\gamma\xi}\mathbf{d}\}+2x_1(\cosh\gamma\pi)\,\mathrm{Re}\,\{(\mathbf{S}_1\mathbf{L}_1^{-1}+i\beta\mathbf{L}_1^{-1})\mathbf{d}\},\\
\mathbf{u}_2 &= -2\sqrt{a^2-x_1^2}\,\mathbf{L}_2^{-1}\,\mathrm{Re}\,\{e^{i\gamma\xi}\mathbf{d}\}+2x_1(\cosh\gamma\pi)\,\mathrm{Re}\,\{(\mathbf{S}_2\mathbf{L}_2^{-1}-i\beta\mathbf{L}_2^{-1})\mathbf{d}\},\\
\mathbf{t}_1^{(1)} &= \frac{-2}{\sqrt{a^2-x_1^2}}\mathbf{G}_3^{(1)}\,\mathrm{Re}\,\{(x_1+2i\gamma a)\,e^{i\gamma\xi}\mathbf{d}\}+2(\cosh\gamma\pi)\,\mathrm{Re}\,\{(\mathbf{G}_1^{(1)}+i\beta\mathbf{G}_3^{(1)})\mathbf{d}\},\\
\mathbf{t}_1^{(2)} &= \frac{2}{\sqrt{a^2-x_1^2}}\mathbf{G}_3^{(2)}\,\mathrm{Re}\,\{(x_1+2i\gamma a)\,e^{i\gamma\xi}\mathbf{d}\}+2(\cosh\gamma\pi)\,\mathrm{Re}\,\{(\mathbf{G}_1^{(2)}-i\beta\mathbf{G}_3^{(2)})\mathbf{d}\},\\
\mathbf{t}_2 &= -\mathbf{t}_\Gamma.
\end{aligned} \tag{47}$$

It is easily verified that (46), (47) reduce to those given in (19), (20) when β (and hence γ) vanishes. From (46) and (47) $\mathbf{u}_1$, $\mathbf{t}_1^{(1)}$ in material 1 are identical to $\mathbf{u}_2$, $\mathbf{t}_1^{(2)}$ in material 2 if $\mathbf{G}_3^{(1)}$, $\mathbf{L}_1^{-1}$ (but not $\mathbf{S}_1\mathbf{L}_1^{-1}$) are replaced by $-\mathbf{G}_3^{(2)}$, $-\mathbf{L}_2^{-1}$, respectively.

The traction $\mathbf{t}_2$ along the interface $|x_1| > a$ as given in (46) depends on $\mathbf{d}$. Consequently $\mathbf{t}_2$ lies on the right eigenplane of $\hat{\mathbf{S}}$ as $\mathbf{t}_\Gamma^\infty$ does (Fig. 1). As x_1 varies from ∞ to a or from $-\infty$ to $-a$, $\mathbf{t}_2$ rotates on the right eigenplane with increasing frequency and amplitude. The crack opening $\Delta\mathbf{u}$ is obtained by subtracting $\mathbf{u}_2$ from $\mathbf{u}_1$ in (47). The second term vanishes due to $(32)_1$ and hence

$$\Delta \mathbf{u} = 2\sqrt{a^2 - x_1^2}\,\mathbf{D}\,\mathrm{Re}\,(e^{i\gamma\xi}\mathbf{d}).$$

Thus $\Delta \mathbf{u}$ depends on $\mathbf{Dd}$ which, according to the discussions in the last section, is on the left eigenplane of $\hat{\mathbf{S}}$ (Fig. 2). As x_1 varies from 0 to a or from 0 to $-a$, $\Delta \mathbf{u}$ rotates on the left eigenplane with increasing frequency and diminishing amplitude.

7. MULTIPLE INTERFACE CRACKS WITH VARIABLE TRACTIONS

The decomposition principle for a single interface crack with constant traction $\mathbf{t}_\Gamma$ can be extended to multiple interface cracks with variable traction $\mathbf{t}_\Gamma(x_1)$ with one minor modification. In Fig. 1 the component $\mathbf{t}_\Gamma^{\gamma}(x_1)$ of $\mathbf{t}_\Gamma(x_1)$ on the right eigenplane may not remain in the same direction for a different x_1. It is therefore necessary to select a fixed right eigenvector $\mathbf{d} = \mathbf{d}_1 + i\mathbf{d}_2$ on the right eigenplane and decompose $\mathbf{t}_\Gamma(x_1)$ in the form

$$\mathbf{t}_\Gamma(x_1) = \mathbf{t}_\Gamma^0(x_1) + \mathbf{t}_\Gamma^{\gamma}(x_1) = t_\Gamma^0(x_1)\mathbf{d}_0 + t_\Gamma^{\gamma}(x_1)\mathbf{d} + \bar{t}_\Gamma^{\gamma}(x_1)\bar{\mathbf{d}}. \tag{48}$$

In the above, $\mathbf{t}_\Gamma^0(x_1)$, $\mathbf{t}_\Gamma^{\gamma}(x_1)$ are real vectors, $t_\Gamma^0(x_1)$ is a real scalar, and $t_\Gamma^{\gamma}(x_1)$, $\bar{t}_\Gamma^{\gamma}(x_1)$ are a pair of complex conjugate scalars. The vectors $\mathbf{d}$, $\bar{\mathbf{d}}$, $\mathbf{d}_0$ are the eigenvectors of $\hat{\mathbf{S}}$ defined in (32).

It is not necessary to solve the eigenrelations (32) to determine $\mathbf{d}$, $\bar{\mathbf{d}}$, $\mathbf{d}_0$. They can be determined explicitly in terms of any real vector as shown in the Appendix. Making use of the orthogonality relations (41), $(48)_2$ yields

$$t_\Gamma^{\gamma}(x_1) = \frac{\bar{\mathbf{d}}^T\mathbf{D}\mathbf{t}_\Gamma(x_1)}{\bar{\mathbf{d}}^T\mathbf{D}\mathbf{d}} = \frac{(\mathbf{d}_1 - i\mathbf{d}_2)\mathbf{D}\mathbf{t}_\Gamma(x_1)}{\mathbf{d}_1^T\mathbf{D}\mathbf{d}_1 + \mathbf{d}_2^T\mathbf{D}\mathbf{d}_2} \tag{49}$$

and $(48)_{1,2}$ furnishes $t_\Gamma^0(x_1)$. The decomposition of $\mathbf{t}_\Gamma(x_1)$ is now complete.

The solution for multiple interface cracks in the bimaterial contains two parts. The first part is

$$\begin{aligned} \mathbf{u}_1 &= \mathrm{Re}\,\{\mathbf{A}_1\langle f_0(z^{(1)})\rangle\mathbf{B}_1^{-1}\}\mathbf{d}_0, \\ \boldsymbol{\phi}_1 &= \mathrm{Re}\,\{\mathbf{B}_1\langle f_0(z^{(1)})\rangle\mathbf{B}_1^{-1}\}\mathbf{d}_0, \end{aligned} \tag{50a}$$

for material 1 in $x_2 > 0$ and

$$\begin{aligned} \mathbf{u}_2 &= \mathrm{Re}\,\{\mathbf{A}_2\langle f_0(z^{(2)})\rangle\mathbf{B}_2^{-1}\}\mathbf{d}_0, \\ \boldsymbol{\phi}_2 &= \mathrm{Re}\,\{\mathbf{B}_2\langle f_0(z^{(2)})\rangle\mathbf{B}_2^{-1}\}\mathbf{d}_0, \end{aligned} \tag{50b}$$

for material 2 in $x_2 < 0$. This is identical to (24a,b) with $\mathbf{t}_\Gamma$ replaced by $\mathbf{d}_0$ and $f_0(z)$ is an unknown function to be determined. The second part is (43a,b) in which $f(z, \gamma)$ is an unknown function to be determined. Let Γ denote the cracks which are located at

$$x_2 = 0, \quad a_k < x_1 < b_k, \quad k = 1, 2, \ldots, n.$$

The functions $f_0(z)$, $f(z, \gamma)$ are continuous except at Γ and vanish at infinity. Satisfaction of conditions (21), (22) and the prescribed tractions at Γ leads to

$$f_0(z) = \bar{f}_0(z), \tag{51}$$

and

$$\begin{aligned} g_0^+(x_1) + g_0^-(x_1) &= -2t_\Gamma^0(x_1), \\ e^{\gamma\pi}g^+(x_1, \gamma) + e^{-\gamma\pi}g^-(x_1, \gamma) &= -2t_\Gamma^{\gamma}(x_1), \end{aligned}$$

$$e^{-\gamma\pi}\bar{g}^{+}(x_1,\gamma)+e^{\gamma\pi}\bar{g}^{-}(x_1,\gamma) = -2\bar{t}_\Gamma(x_1), \tag{52}$$

where

$$g_0(z) = \frac{d}{dz}f_0(z), \quad g(z,\gamma) = \frac{\partial}{\partial z}f(z,\gamma).$$

Equations (52) are the Hilbert problem for which the solutions are (Muskhelishvili, 1953)

$$g_0(z) = -\frac{\chi(z,0)}{\pi i}\int_\Gamma \frac{t_\Gamma^0(\lambda)\,d\lambda}{\chi^+(\lambda,0)(\lambda-z)} + P_0(z)\chi(z,0),$$

$$g(z,\gamma) = -\frac{\chi(z,\gamma)}{\pi i}\int_\Gamma \frac{e^{-\gamma\pi}t_\Gamma(\lambda)\,d\lambda}{\chi^+(\lambda,\gamma)(\lambda-z)} + P(z,\gamma)\chi(z,\gamma),$$

$$\bar{g}(z,\gamma) = -\frac{\bar{\chi}(z,\gamma)}{\pi i}\int_\Gamma \frac{e^{\gamma\pi}\bar{t}_\Gamma(\lambda)\,d\lambda}{\bar{\chi}^+(\lambda,\gamma)(\lambda-z)} + \bar{P}(z,\gamma)\bar{\chi}(z,\gamma).$$

In the above

$$\chi(z,\gamma) = \prod_{k=1}^{n}(z-a_k)^{-1/2-i\gamma}(z-b_k)^{-1/2+i\gamma}$$

and $P_0(z)$, $P(z,\gamma)$ are polynomials in z of order less than n. They are to be determined such that the crack opening at all crack tips vanishes. It can be shown that

$$i e^{\gamma\pi}\chi^+(\lambda,\gamma) \quad \text{and} \quad i e^{-\gamma\pi}\bar{\chi}^+(\lambda,\gamma)$$

are complex conjugates of each other, confirming that the functions $g(z,\gamma)$ and $\bar{g}(z,\gamma)$ are indeed complex conjugates of each other. Likewise, the solution for $g_0(z)$ satisfies condition (51).

The traction along the interface is, by substituting (50a,b) and (43a,b) into (18),

$$\mathbf{t}_2(x_1) = g_0(x_1)\mathbf{d}_0 + 2(\cosh\gamma\pi)\,\mathrm{Re}\,\{g(x_1,\gamma)\mathbf{d}\}.$$

The traction from the first term is in the direction of the right null vector of $\hat{\mathbf{S}}$ while that from the second term lies on the right eigenplane. By writing

$$f_0^+(x_1) = [f_0^+(x_1)-f_0^-(x_1)]+f_0^-(x_1)$$

and similar expressions for $f^+(x_1,\gamma)$, $\bar{f}^+(x_1,\gamma)$, the crack opening displacement $\Delta\mathbf{u}$ can be shown to be

$$\Delta\mathbf{u} = -\tfrac{1}{2}(\mathbf{W}+i\mathbf{D})\{[f_0^+(x_1)-f_0^-(x_1)]\mathbf{d}_0+e^{\gamma\pi}[f^+(x_1,\gamma)-f^-(x_1,\gamma)]\mathbf{d} + e^{-\gamma\pi}[\bar{f}^+(x_1,\gamma)-\bar{f}^-(x_1,\gamma)]\bar{\mathbf{d}}\}. \tag{53}$$

Observing that the complex conjugates of

$$f_0^+(x_1), \quad f^+(x_1,\gamma), \quad f^-(x_1,\gamma)$$

are, respectively,

$$f_0^-(x_1), \quad \bar{f}^-(x_1,\gamma), \quad \bar{f}^+(x_1,\gamma)$$

and setting

$$\mathbf{W}+i\mathbf{D} = \mathbf{D}(\hat{\mathbf{S}}+i\mathbf{I}),$$

equation (53) can be further simplified by using (32) as

$$\Delta\mathbf{u} = \mathrm{Im}\,\{f_0^+(x_1)\}\mathbf{D}\mathbf{d}_0+\sqrt{1-\beta^2}\,\mathrm{Im}\,\{[f^+(x_1,\gamma)-f^-(x_1,\gamma)]\mathbf{D}\mathbf{d}\}.$$

It is clear that the first term is in the direction of the left null vector of $\hat{\mathbf{S}}$ while the second terms lie on the left eigenplane.

8. CONCLUDING REMARKS

The oscillatory field in displacement leads to a physically unrealistic phenomenon of interpenetration of the crack surfaces, although the region of interpenetration is generally small. There have been several studies on the problem to eliminate the unrealistic interpenetration; see, for example, Comninou (1977), Comninou and Schmueser (1979), Achenbach *et al.* (1979), Knowles and Sternberg (1983) and Gautesen and Dundurs (1988). The Comninou model of partial opening of the interface crack in isotropic bimaterials has been extended to anisotropic bimaterials by Wang and Choi (1983), Wu and Hwang (1990) and Ni and Nemat-Nasser (in press).

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APPENDIX: AN EXPLICIT SOLUTION OF THE EIGENVECTORS OF $\hat{\mathbf{S}}$

The matrix $\hat{\mathbf{S}}$ defined in (31) has the properties (Ting, 1986; Chadwick and Ting, 1987):

$$\operatorname{tr}\hat{\mathbf{S}} = 0, \quad \det\hat{\mathbf{S}} = 0.$$

Therefore the equation for the eigenvalues λ is

$$\lambda^3 + \beta^2\lambda = 0, \tag{A1}$$

where β is given in (30). The eigenvalues are $-i\beta$, $i\beta$, 0 and the associated eigenvectors $\mathbf{d}$, $\bar{\mathbf{d}}$, $\mathbf{d}_0$ are related by (32). To find $\mathbf{d}$ and $\mathbf{d}_0$, take any non-zero real vector $\mathbf{t}$ and let

$$\beta^2\mathbf{t} = \mathbf{d}_0 + \mathbf{d} + \bar{\mathbf{d}}. \tag{A2}$$

Multiplying by $\hat{\mathbf{S}}$, $\hat{\mathbf{S}}^2$, and employing (32) leads to

$$\begin{aligned} \beta\hat{\mathbf{S}}\mathbf{t} &= -i(\mathbf{d} - \bar{\mathbf{d}}), \\ \hat{\mathbf{S}}^2\mathbf{t} &= -(\mathbf{d} + \mathbf{d}). \end{aligned} \tag{A3}$$

Hence

$$\mathbf{d} = -\tfrac{1}{2}(\hat{\mathbf{S}} - i\beta\mathbf{I})\hat{\mathbf{S}}\mathbf{t} \tag{A4}$$

and from (A2), $(\text{A3})_2$,

$$\mathbf{d}_0 = (\hat{\mathbf{S}}^2 + \beta^2\mathbf{I})\mathbf{t}. \tag{A5}$$

With $\mathbf{d}_1$, $\mathbf{d}_2$ denoting the real and imaginary parts of $\mathbf{d}$ (33), an alternative expression obtained from (A4), (A2) is

$$\mathbf{d}_2 = \tfrac{1}{2}\beta\hat{\mathbf{S}}\mathbf{t}, \quad \mathbf{d}_1 = -\frac{1}{\beta}\hat{\mathbf{S}}\mathbf{d}_2, \quad \mathbf{d}_0 = \beta^2\mathbf{t}-2\mathbf{d}_1. \tag{A6}$$

Similar expressions and different expressions have been obtained by Wu (in press) and Gao *et al.* (in press), respectively.

Equations (A4)–(A6) provide an explicit expression for $\mathbf{d}$, $\mathbf{d}_0$ in terms of an arbitrarily chosen real vector $\mathbf{t}$. If $\mathbf{t}$ happens to be proportional to $\mathbf{d}_0$, $\mathbf{d}$ obtained from (A4) or $(A6)_{1,2}$ vanishes. Likewise, if $\mathbf{t}$ is on the right eigenplane, $\mathbf{d}_0$ obtained from (A5) or $(A6)_3$ vanishes. In these cases a different $\mathbf{t}$ should be employed. $\mathbf{d}$ obtained from (A4) for different choices of $\mathbf{t}$ differ by a complex multiplier.

According to Cayley–Hamilton principle (A1) applies to $\hat{\mathbf{S}}$, i.e.

$$\hat{\mathbf{S}}^3+\beta^2\hat{\mathbf{S}} = \mathbf{0}.$$

This confirms that $\mathbf{d}$, $\mathbf{d}_0$ of (A4), (A5) indeed satisfy $(32)_{1,3}$ for arbitrary $\mathbf{t}$.